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Abstract

We consider a one-dimensional reflection problem for a stack of dielectric slabs with dielectric constants which are independent samples of a Bernoulli random variable. The widths of all the slabs are the same, and each slab extends to infinity in any horizontal direction. A plane wave of unit amplitude is normally incident on the stack from below. The quantities to be calculated are the average reflected and transmitted powers. We estimate the error committed in these calculations when multiple scattering is neglected; in particular, we investigate the effect of the random nature of the scattering medium in diminishing this error. A detailed analysis of the various possible cases is given, and a number of relevant curves and tables are presented.

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1. Introduction

In treating problems involving the scattering of incident radiation (e.g., sound waves, radio waves) by a random medium, it has long been recognized that the random nature of the scattering medium plays an important part in attenuating multiple scattering. A familiar qualitative explanation of this phenomenon is that the internal field produced in the medium by single scattering is considerably weakened by the "incoherence", or, to put it differently, that the contributions to the internal field from uncorrelated parts of the medium add like power rather than like amplitude. However, with the exception of the work of Mintzer [1], there appears to be little in the way of explicit calculations of the strength of multiple scattering in a random medium. The difficulty seems to stem mainly from the three-dimensional nature of the usual scattering problems, which requires one to carry out difficult multiple integrations in order to estimate the high order scattering.

In the present paper, we treat the multiple scattering of incident plane waves by a random stack of dielectric slabs. Because of the one-dimensional nature of the problem, and because of the simple way in which the random stack is generated, we are able to carry the analysis of the effects of multiple scattering quite far. In particular, we find that we can get an upper bound for the strength of scattering of any order which, unlike the crude estimates usually given, takes into account the random nature of the scattering medium. Our basic result is that when the number of slabs is large, the Neumann series for the mean square transmission and reflection coefficients converge much more rapidly than would be inferred from the smallness of the perturbation of the incident field by the scattering

medium. This additional attenuation of multiple scattering is directly attributable to the random nature of the scattering medium. However, it is found that the randomness of the scattering medium is no longer effective in reducing multiple scattering when the order of the scattering exceeds the number of slabs in the stack.

Despite the evident difficulty of extending the analysis to three-dimensional scattering problems and more complicated scattering media, there is every reason to believe that our qualitative conclusions are quite generally valid.

2. Formulation of the problem

Consider a stack of N dielectric slabs, all of the same thickness d. Choose the origin of a Cartesian system in the lower face of the bottom slab, and let the z-axis point upwards in the direction normal to the slabs.* The configuration is illustrated in Figure 1.

The random function $\epsilon(z)$ which describes the dielectric constant profile of the stack of N slabs is constructed as follows. Let β be a Bernoulli random variable, which assumes the values -1 and +1 with equal probability, i.e., with probability 1/2. To obtain a realization (sample function) of the random function $\epsilon(z)$ we perform a sequence of N <u>independent</u> trials, each consisting of an observation of β, and we denote the outcomes of these trials by $\beta(1,\alpha)$, $\beta(2,\alpha)$, ..., $\beta(N,\alpha)$. (The index α ranges over a sample space S consisting of 2^N points, each of which corresponds to a possible outcome of the sequence of N trials.) Then the realization indexed by α of the random function $\epsilon(z)$ is given by the piecewise constant function**

---

* The slabs are regarded as infinite in the x and y directions.

** To avoid confusion, note that Δ is used exclusively to designate an increment of the quantity it precedes, and is never used as an independent symbol.

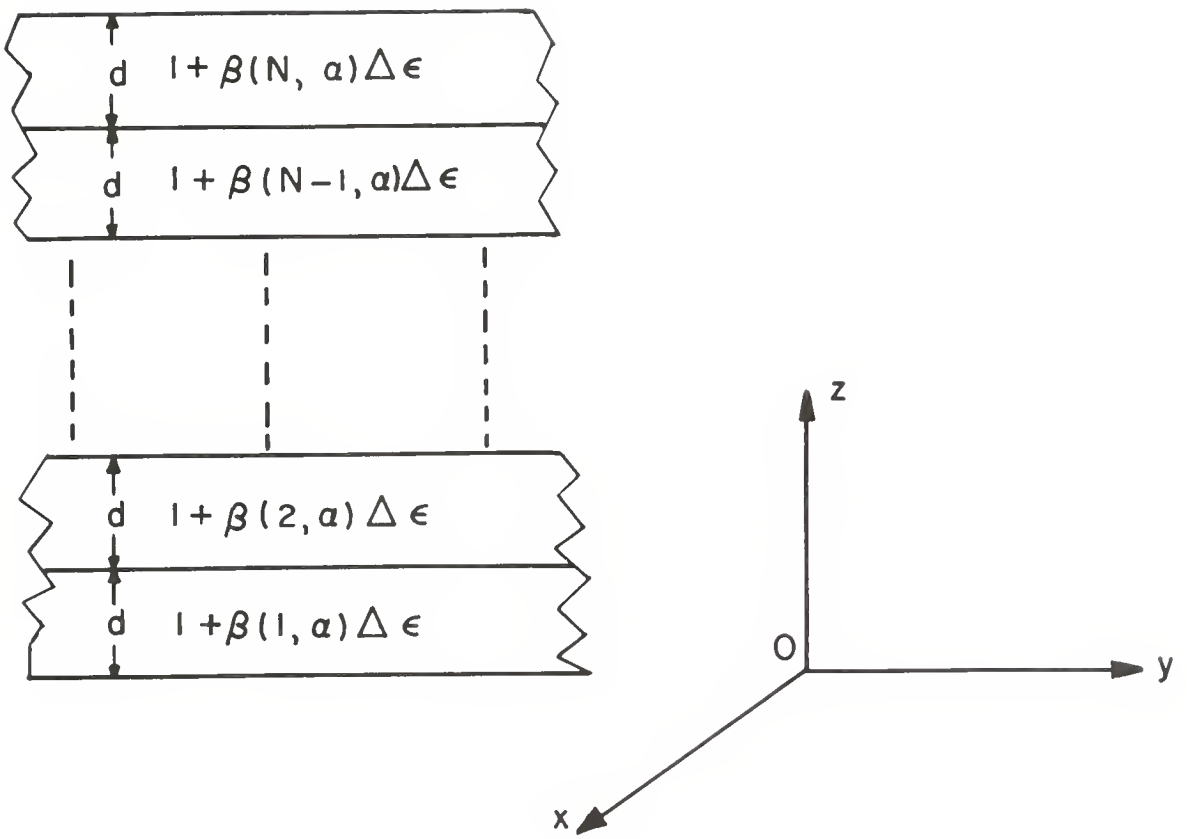

Figure 1. The scattering medium (schematic).

$$(1) \qquad \epsilon(z,\alpha) = 1 + \Delta\epsilon(z,\alpha) = \begin{cases} 1 & , \ z < 0 \\ 1+\beta(r,\alpha)\Delta\epsilon & , \ (r-1)d \le z < rd, \ 1 \le r \le N, \ \Delta\epsilon > 0. \\ 1 & , \ z \ge Nd \end{cases}$$

For example, one might toss a well-balanced coin and assign one of the slabs the dielectric constant $1 + \Delta\epsilon$ if the outcome is a head, and $1 - \Delta\epsilon$ if the outcome is a tail, then toss the coin again and assign any other slab the dielectric constant $1 + \Delta\epsilon$ if the outcome is a head and $1 - \Delta\epsilon$ if the outcome is a tail, and so on until all N slabs have been assigned values of the dielectric constant. We shall often indulge in the usual abuse of notation, whereby we suppress the index α and use the same symbol to denote both a random variable and the values it assumes, or to denote a random function and its realizations.

Suppose now that a scalar plane wave e^{ikz} is incident from below on a random stack of slabs generated in the manner just described, and that the resulting field $u(z)$ obeys the inhomogeneous wave equation*

$$(2) \qquad \left[\frac{d^2}{dz^2} + k^2\right] u(z) = - k^2 \Delta\epsilon(z)u(z).$$

Since $\Delta\epsilon(z)$ is a random function, the same is true of $u(z)$; the index α has been suppressed, so that (2) can be regarded either as a stochastic differential equation, or as the wave equation obeyed separately by each realization of $u(z)$. The integral equation corresponding to (2) is

$$(3) \qquad u(z) = e^{ikz} - k^2 \int_0^{Nd} G(z,z')\,\Delta\epsilon(z')u(z')dz', \qquad \text{where}$$

---

* It is also assumed that $u(z)$ and $du(z)/dz$ are continuous. For simplicity we have chosen the dielectric constant of the medium in which the stack is immersed to be unity, with the result that $\epsilon(z,\alpha)$ can be less than unity. Those who are disturbed by this can replace $\epsilon(z,\alpha)$ in (1) by $\epsilon(z,\alpha)/\epsilon_o$, where $\epsilon_o > 1$ and $\epsilon_o(1-\Delta\epsilon) \ge 1$; equation (2) remains the same.

$$G(z,z') = \frac{1}{2ik}\, e^{ik|z-z'|}$$

is the one-dimensional free-space Green's function, and the incident wave e^{ikz} is, of course, a solution of the homogeneous wave equation. In writing the limits of integration, we have used the fact that by construction the random function $\Delta\epsilon(z)$, which is the dielectric constant profile centered about its mean, vanishes outside of the interval $0 \leq z < Nd$. The formal solution of (3) is given by the Neumann series

$$\begin{aligned} u(z) &= e^{ikz} + \frac{ik}{2}\int_0^{Nd} e^{ik|z-z_1|} e^{ikz_1}\,\Delta\epsilon(z_1)dz_1 \\ &\quad - \frac{k^2}{4}\int_0^{Nd}\int_0^{Nd} e^{ik|z-z_1|} e^{ik|z_1-z_2|} e^{ikz_2}\Delta\epsilon(z_1)\Delta\epsilon(z_2)dz_1dz_2 + \cdots \\ &= 1 + \sum_{n=1}^{\infty}\left(\frac{ik}{2}\right)^n \int_0^{Nd}\int_0^{Nd}\cdots\int_0^{Nd} e^{ik|z-z_1|} e^{ik|z_1-z_2|} \cdots e^{ik|z_{n-1}-z_n|} e^{ikz_n} \\ &\qquad \Delta\epsilon(z_1)\Delta\epsilon(z_2)\cdots\Delta\epsilon(z_n)dz_1dz_2\cdots dz_n, \end{aligned} \tag{4}$$

where, for the moment, we leave aside the question of convergence.

It is particularly convenient in one-dimensional problems of this kind to describe the solution u(z) in terms of a transmission coefficient T and a reflection coefficient R, defined by the relations

$$\begin{aligned} u(z) &= T\, e^{ikz} \qquad\qquad z \geq Nd\,, \\ u(z) &= e^{ikz} + R\, e^{-ikz}\,, \qquad z < 0\,. \end{aligned} \tag{5}$$

It follows from (4) and (5) that T and R are given by

$$T = 1 + \frac{ik}{2}\int_0^{Nd} \Delta\epsilon(z_1)dz_1 - \frac{k^2}{4}\int_0^{Nd}\int_0^{Nd} e^{-ikz_1} e^{ik|z_1-z_2|} e^{ikz_2}$$

$$\Delta\epsilon(z_1)\Delta\epsilon(z_2)dz_1dz_2 + \cdots \tag{6}$$

$$= 1 + \sum_{n=1}^{\infty} \left(\frac{ik}{2}\right)^n \int_0^{Nd}\int_0^{Nd}\cdots\int_0^{Nd} e^{-ikz_1} e^{ik|z_1-z_2|}\cdots e^{ik|z_{n-1}-z_n|} e^{ikz_n}$$

$$\Delta\epsilon(z_1)\Delta\epsilon(z_2)\cdots\Delta\epsilon(z_n)dz_1dz_2\cdots dz_n,$$

and

$$R = \frac{ik}{2}\int_0^{Nd} e^{2ikz_1}\Delta\epsilon(z_1)dz_1 - \frac{k^2}{4}\int_0^{Nd}\int_0^{Nd} e^{ikz_1} e^{ik|z_1-z_2|} e^{ikz_2}$$

$$\Delta\epsilon(z_1)\Delta\epsilon(z_2)dz_1dz_2 + \cdots \tag{7}$$

$$= \sum_{n=1}^{\infty} \left(\frac{ik}{2}\right)^n \int_0^{Nd}\int_0^{Nd}\cdots\int_0^{Nd} e^{ikz_1} e^{ik|z_1-z_2|}\cdots e^{ik|z_{n-1}-z_n|} e^{ikz_n}$$

$$\Delta\epsilon(z_1)\Delta\epsilon(z_2)\cdots\Delta\epsilon(z_n)dz_1dz_2\cdots dz_n,$$

respectively. We remind the reader that T and R are random variables, a fact which may be emphasized by writing $T(\alpha)$ and $R(\alpha)$, i.e., by explicitly indicating the realization index α. Since the random function $\Delta\epsilon(z)$ is constant in each slab, the multiple integrals appearing in (6) and (7) reduce to sums. Thus, using (1), we can write (6) as

$$T(\alpha) = 1 + \sum_{n=1}^{\infty} \left(\frac{k\Delta\epsilon}{2}\right)^n \sum_{r_1,r_2,\cdots,r_n=1}^{N} a(r_1,r_2,\ldots,r_n)\beta(r_1,\alpha)\beta(r_2,\alpha)\ldots\beta(r_n,\alpha), \tag{8}$$

where the coefficients $a(r_1,r_2,\ldots,r_n)$ are defined by

$$(9)\quad a(r_1,r_2,\ldots,r_n) = i^n \int_{(r_1-1)d}^{r_1 d} \int_{(r_2-1)d}^{r_2 d} \cdots \int_{(r_n-1)d}^{r_n d} e^{-ikz_1} e^{ik|z_1-z_2|} \cdots e^{ik|z_n-z_{n-1}|} e^{ikz_n} \, dz_1 dz_2 \cdots dz_n .$$

The summation with respect to the indices r_1, r_2, ..., r_n extends over all the N^n distinct n-tuples $(r_1,r_2,\ldots,r_n)$. Similarly, we can write (7) as

$$(10)\quad R(\alpha) = \sum_{n=1}^{\infty} \left(\frac{k\Delta\epsilon}{2}\right)^n \sum_{r_1,r_2,\ldots,r_n=1}^{N} b(r_1,r_2,\ldots,r_n)\beta(r_1,\alpha)\beta(r_2,\alpha)\ldots\beta(r_n,\alpha),$$

where the coefficients $b(r_1,r_2,\ldots,r_n)$ are defined by

$$(11)\quad b(r_1,r_2,\ldots,r_n) = i^n \int_{(r_1-1)d}^{r_1 d} \int_{(r_2-1)d}^{r_2 d} \cdots \int_{(r_n-1)d}^{r_n d} e^{ikz_1} e^{ik|z_1-z_2|} \cdots e^{ik|z_{n-1}-z_n|} e^{ikz_n} \, dz_1 dz_2 \ldots dz_n .$$

It is important to note that

$$(12)\qquad |a(r_1,r_2,\ldots,r_n)| \le d^n, \qquad |b(r_1,r_2,\ldots,r_n)| \le d^n,$$

which follows at once from the fact that the integrands in (9) and (11) are complex quantities with unit modulus.

Turning to the question of the convergence of the series (8) and (10), we note the inequalities

$$|T(\alpha)| \le 1 + \sum_{n=1}^{\infty} \left(\frac{kd\Delta\epsilon}{2}\right)^n \sum_{r_1,\ldots,r_n} |\beta(r_1,\alpha)\cdots\beta(r_n,\alpha)| = 1 + \sum_{n=1}^{\infty} \left(\frac{kNd\Delta\epsilon}{2}\right)^n,$$

$$|R(\alpha)| \le \sum_{n=1}^{\infty} \left(\frac{kd\Delta\epsilon}{2}\right)^n \sum_{r_1,\ldots,r_n} |\beta(r_1,\alpha)\cdots\beta(r_n,\alpha)| = \sum_{n=1}^{\infty} \left(\frac{kNd\Delta\epsilon}{2}\right)^n,$$

where we have used (12). It follows at once that the Neumann series (8) and (10) are absolutely convergent if

$$A = \frac{kd\Delta\epsilon}{2} < \frac{1}{N} \quad . \tag{13}$$

It should be noted that the convergence condition (13) guarantees the convergence of the Neumann series for $T(\alpha)$ and $R(\alpha)$ for <u>all</u> α in S.

In the remainder of this paper we shall be concerned with the mean square values of $T(\alpha)$ and $R(\alpha)$, since these are the quantities of greatestpractical interest. Therefore, we now convert the Neumann series (8) and (10) into corresponding Neumann series for $<|T|^2>$ and $<|R|^2>$. By the angular brackets we mean the <u>ensemble average</u>, i.e., the average over all realizations of $\Delta\epsilon(z)$. More explicitly, if $f(\alpha)$ is a function of the point α of the (discrete) sample space S, then

$$< f(\alpha) > = \sum_{\alpha \text{ in } S} p(\alpha) f(\alpha),$$

where $p(\alpha)$ is the probability of the event corresponding to α; in our case, S is the space consisting of the 2^N possible outcomes of a sequence of N independent Bernoulli trials, and $p(\alpha)$ has the value 2^{-N} for every α in S. Thus, multiplying (8) and (10) by their complex conjugates and ensemble averaging, we find

$$<|T|^2> = 1 + \sum_{n=1}^{\infty} \left(\frac{k\Delta\epsilon}{2}\right)^{2n} \sum_{r_1,\ldots,r_{2n}} \left[a(r_1,\ldots,r_{2n}) + a^*(r_1,\ldots,r_{2n})\right]$$

$$< \beta(r_1)\cdots\beta(r_{2n}) > \tag{14}$$

$$+ \sum_{n=1}^{\infty} \left(\frac{k\Delta\epsilon}{2}\right)^{2n} \sum_{\substack{\ell+m=2n \\ \ell,m>0}} \sum_{r_1,\ldots,r_\ell} \sum_{s_1,\ldots,s_m} a(r_1,\ldots,r_\ell)a^*(s_1,\ldots,s_m)$$

$$< \beta(r_1)\cdots\beta(r_\ell)\beta(s_1)\cdots\beta(s_m) >,$$

and

$$<|R|^2> = \sum_{n=1}^{\infty} \left(\frac{k\Delta\epsilon}{2}\right)^{2n} \sum_{\substack{\ell+m=2n \\ \ell,m>0}} \sum_{r_1,\ldots,r_\ell}$$

(15)
$$\sum_{s_1,\ldots,s_m} b(r_1,\ldots,r_\ell)b^*(s_1,\ldots,s_m)$$
$$< \beta(r_1)\cdots\beta(r_\ell)\beta(s_1)\cdots\beta(s_m) >, (\ell,m > 0),$$

where we suppress the index α, and make use of the fact that averages of products of an odd number of the random variables $\beta(r)$ vanish. It is easily verified that the sum of the terms in (14) or (15) which are of a given order in $\Delta\epsilon$ are real. Since

(16)
$$< \beta(r_1)\cdots\beta(r_\ell)\beta(s_1)\cdots\beta(s_m) > \leq 1,$$

we have the inequalities

(17)
$$<|T|^2> \leq 1 + \sum_{n=1}^{\infty} (2n+1)\left(\frac{kNd\Delta\epsilon}{2}\right)^{2n},$$

and

(18)
$$<|R|^2> \leq \sum_{n=1}^{\infty} (2n-1)\left(\frac{kNd\Delta\epsilon}{2}\right)^{2n}.$$

Just as one would expect, these series converge if and only if the condition (13) is satisfied.

We shall now show that, because of the random nature of the scattering medium, the convergence of the series (14) and (15) is usually more rapid than one would infer from the convergence condition (13). To show this, we must first prove an auxiliary inequality.

Basic inequality

Let $c(r_1,r_2,\ldots,r_n)$ stand for either the coefficient $a(r_1,r_2,\ldots,r_n)$ defined by (9) or the coefficient $b(r_1,r_2,\ldots,r_n)$ defined by (11). Let $\beta(r)$, $1 \leq l \leq N$,

denote N independent random variables, each of which takes the values -1 and +1 with equal probability. Then, if $\ell, m > 0$ and $\ell + m = 2n$,

$$(19) \qquad \left| \sum_{r_1,\ldots,r_\ell} \sum_{s_1,\ldots,s_m} c(r_1,\ldots,r_\ell)c^*(s_1,\ldots,s_m) < \beta(r_1)\cdots\beta(r_\ell)\beta(s_1)\cdots\beta(s_m) > \right| \leq d^n \left\langle \left[\sum_{r=1}^{N} \beta(r) \right]^{2n} \right\rangle .$$

Proof

Writing the quantity

$$(20) \qquad < \beta(r_1)\cdots\beta(r_\ell)\beta(s_1)\cdots\beta(s_m) >$$

in the form $< \beta^{e_1}(1)\beta^{e_2}(2)\ldots\beta^{e_N}(N) >$, where $e_1, e_2, \ldots, e_N$ are suitable non-negative integers with sum 2n, we see that (20) is non-negative, since it either vanishes (if any of the e_i is odd) or is positive (if none of the e_i is odd). Moreover, by (12), the coefficients $c(r_1,\ldots,r_\ell)c^*(s_1,\ldots,s_m)$ appearing in (19) all have absolute values $\leq d^{2n}$. The inequality (19) then follows from the obvious fact that if A, B,...,X are non-negative and if $|a|, |b|, \ldots, |x| \leq C$, then

$$|aA + bB + \ldots + xX| \leq C(A + B + \ldots + X),$$

and from the identity

$$\sum_{r_1,\ldots,r_\ell} \sum_{s_1,\ldots,s_m} < \beta(r_1)\ldots\beta(r_\ell)\beta(s_1)\ldots\beta(s_m) > = \left\langle \left[\sum_{r=1}^{N} \beta(r) \right]^{2n} \right\rangle ,$$

$$\ell + m = 2n.$$

Finally, we note that the inequality

$$(21) \qquad \left| \sum_{r_1,\ldots,r_{2n}} a(r_1,\ldots,r_{2n}) < \beta(r_1)\ldots\beta(r_{2n}) > \right| \leq d^{2n} \left\langle \left[\sum_{r=1}^{N} \beta(r) \right]^{2n} \right\rangle$$

can be proved in just the same way.

3. The stochastic majorants

The qualitative idea behind the considerations of this section is the following: As shown by (13), the requirement $AN = kNd\,\Delta\epsilon/2 < 1$ is sufficient to guarantee the convergence of the Neumann series (14) and (15) for $<|T|^2>$ and $<|R|^2>$. On the other hand, it is clear that AN is not an appropriate quantity to use in estimating the rate of convergence of the series (14) and (15). To see this, we observe that we have not taken into account the random nature of the scattering medium in deriving (17) and (18), except, of course, insofar as we have used the fact that expectations of products of an odd number of $\beta(r)$ vanish. Indeed, by writing

$$(16) \qquad < \beta(r_1) \cdots \beta(r_\ell)\beta(s_1) \cdots \beta(s_m) > \; \leq \; 1,$$

we have automatically eliminated any effect that the statistics of the problem might have in attenuating high order scattering. Since, when N is large, most of the realizations of $\Delta\epsilon(z)$ exhibit many changes in sign, one might expect, to take a rather extreme example, that the internal field produced in the stack of slabs by single scattering is less than what it would be if the profile had the constant value $\Delta\epsilon$ throughout the stack of slabs*. The rather loose phrases "coherent scattering" and "incoherent scattering" are ordinarily used to distinguish between these two situations. We shall now apply the lemmas of the preceding section to derive majorant series for the absolute value of the error committed in omitting terms of order $\geq 2n_o$ in the series (14) and (15) for $<|T|^2>$ and $<|R|^2>$. These majorants take into account the random nature of the scattering medium, i.e., the oscillatory nature or "wiggliness" of most of the realizations of $\Delta\epsilon(z)$, and will accordingly be called stochastic majorants. We shall find that, although the condition $AN < 1$ is still needed to ensure the convergence of the stochastic majorant series, a fact which is

*In our case, there is probability 2^{-N} of this occurring.

perhaps surprising at first, the rate of convergence of the stochastic majorants is usually much faster than would be inferred from the size of AN.

The derivation of the stochastic majorants follows almost immediately when, instead of the crude inequality (16), we use the more refined inequalities (19) and (21), proved in the preceding section. Thus, except for the factor $(k\Delta\epsilon/2)^{2n}$, the term of order 2n in the sum (14) is the sum of

$$(22) \qquad \sum_{r_1,\ldots,r_{2n}} \left[a(r_1,\ldots,r_{2n}) + a^*(r_1,\ldots,r_{2n})\right] < \beta(r_1)\ldots\beta(r_{2n}) >$$

and

$$(23) \qquad \sum_{\substack{\ell+m=2n \\ \ell,m>0}} \sum_{r_1,\ldots,r_\ell} \sum_{s_1,\ldots,s_m} a(r_1,\ldots,r_\ell)a^*(s_1,\ldots,s_m) < \beta(r_1)\ldots\beta(r_\ell)\beta(s_1)\ldots\beta(s_m) >.$$

Using (21), we see that the absolute value of (22) is bounded above by

$$2d^{2n} \left\langle \left[\sum_{r=1}^{N} \beta(r) \right]^{2n} \right\rangle.$$

Moreover, using (19), we see that the absolute value of (23) is bounded above by

$$(2n-1)d^{2n} \left\langle \left[\sum_{r=1}^{N} \beta(r) \right]^{2n} \right\rangle .$$

It follows at once from the form of (14) that if $E_T(2n_o)$ is the error committed in neglecting terms in (14) with orders $\geq 2n_o$, then

$$(24) \qquad \left| E_T(2n_o) \right| \leq \sum_{n=n_o}^{\infty} (2n+1) \left(\frac{kd\Delta\epsilon}{2} \right)^{2n} < \beta_N^{2n} >, \quad 0 \leq n_o < \infty,$$

where the zero-mean random variable β_N is defined by

$$(25) \qquad \beta_N = \sum_{r=1}^{N} \beta(r),$$

i.e., β_N is the sum of N independent random variables, each of which takes on the values -1 and 1 with equal probability. The distribution of β_N is, of

course, the familiar binomial distribution of probability theory, for which

$$(26) \qquad \text{Prob}\left\{\beta_N = N - 2r\right\} = 2^{-N}\binom{N}{r}, \qquad 0 \le r \le N.$$

The difference between the right side of (24) and the corresponding terms of (17) is that we have

$$(27) \qquad < \beta_N^{2n} > \; = 2^{-N} \sum_{r=0}^{N} \binom{N}{r}(N - 2r)^{2n}$$

in place of N^{2n}. Thus, the crudeness of the estimate (17) consists in replacing $< \beta_N^{2n} >$ by the value N^{2n}, i.e., in failing to take account of the fact that the random variable β_N^{2n} takes on its maximum value of N^{2n} only with the probability 2^{1-N}, which is small when the number of slabs is large.

In just the same way, using (15), (19) and (21), we find that if $E_R(2n_o)$ is the error committed in neglecting terms in (15) of order $\ge 2n_o$, then

$$(28) \qquad \left|E_R(2n_o)\right| \le \sum_{n=n_o}^{\infty} (2n-1)\left(\frac{kd\Delta\epsilon}{2}\right)^{2n} < \beta_N^{2n} >, \qquad 1 \le n_o < \infty,$$

which again differs from the corresponding terms of (18) only in the important respect of having $< \beta_N^{2n} >$ in place of N^{2n}. Since the majorant series (17) and (18) do not take account of the random nature of the scattering medium, we shall refer to them for brevity as the <u>sure</u> majorants, using the word <u>sure</u> as an antonym for <u>stochastic</u>.

4. <u>The binomial moments</u>

To proceed with the analysis of the stochastic majorants, we must first study the behavior of the binomial moment $< \beta_N^{2n} >$ as a function of N, the number of slabs, and 2n, the order of the moment (the odd moments vanish). For convenience, we introduce the <u>scaled</u> random variable

$$\hat{\beta}_N = A\,\beta_N,$$

and deal with the corresponding <u>scaled</u> binomial moment

$$< \hat{\beta}_N^{2n} > \; = \; <(A\beta_N)^{2n} > = \; <\left(\frac{kd\Delta\epsilon}{2}\;\beta_N\right)^{2n} >,$$

rather than $< \beta_N^{2n} >$. It follows from $<(A\beta_N)^{2n} > \leq (AN)^{2n}$ and $AN < 1$ that $< \hat{\beta}_N^{2n} >$ approaches zero with increasing n, unlike $< \beta_N^{2n} >$. By (27), we have

$$(29) \qquad < \hat{\beta}_N^{2n} > \; = \; 2^{-N} \sum_{r=0}^{N} \binom{N}{r} \left[A(N-2r)\right]^{2n},$$

obviously a monotone decreasing function of 2n. It is to be expected that the behavior of $< \hat{\beta}_N^{2n} >$ depends on the relative size of 2n and N.

Qualitatively, the conclusions which we shall establish in this section are the following: When $2n \ll N$, $< \hat{\beta}_N^{2n} >$ can be estimated by the corresponding (scaled) moment $< g_N^{2n} >$ of g_N, the zero-mean Gaussian random variable with the same variance; this is reasonable in view of the central limit theorem of probability theory. As 2n approaches N, i.e., as the order of the moment approaches the number of slabs, this approximation breaks down completely, and we must use (29) to find $< \hat{\beta}_N^{2n} >$. Indeed, for fixed N, the quantity $< g_N^{2n} >$ instead of going to zero becomes infinite as n increases. When $2n \gg N$, the scaled binomial moment $< \hat{\beta}_N^{2n} >$ can be estimated by the quantity $2^{1-N}(AN)^{2n}$, which goes to zero exponentially as n increases. However, as is apparent from this exponential estimate, the condition $AN < 1$ is still needed to ensure the convergence even of the stochastic majorants.

We now discuss these matters in more detail.

a. <u>The Gaussian estimate of</u> $< \hat{\beta}_N^{2n} >$

To obtain the Gaussian estimate of $< \hat{\beta}_N^{2n} >$, we consider the zero-mean Gaussian random variable g_N with the same variance as $\hat{\beta}_N$, i.e., with variance A^2N. The Gaussian moments are given by [2]

$$(30) \qquad < g_N^{2n} > \; = 1 \cdot 3 \cdots (2n-1) A^{2n} N^n,$$

while all odd moments of g_N vanish. Moreover, we have the inequality

$$(31) \qquad < \hat{\beta}_N^{2n} > \; \leq \; < g_N^{2n} >, \qquad 1 \leq N < \infty, \quad 0 \leq n < \infty,$$

which means that the Gaussian moments are upper bounds for the binomial moments. To show this, we first observe that

$$\hat{\beta}_N = \sum_{r=1}^{N} \hat{\beta}(r),$$

where $\hat{\beta}(r)$, $1 \leq r \leq N$, is a sequence of independent Bernoulli random variables, each of which takes the values -A and +A with equal probability. We then write

$$g_N = \sum_{r=1}^{N} g(r),$$

where $g(r)$, $1 \leq r \leq N$, is a sequence of independent Gaussian random variables, each with mean zero and variance A^2. It follows by the multinomial expansion that

$$(32) \quad < \hat{\beta}_N^{2n} > = \sum_{m_1,\ldots,m_N} \frac{(2n)!}{m_1!\ldots m_N!} < \hat{\beta}^{m_1}(1) > \cdots < \hat{\beta}^{m_N}(N) > ,$$

$$(33) \quad < g_N^{2n} > = \sum_{m_1,\ldots,m_N} \frac{(2n)!}{m_1!\ldots m_N!} < g^{m_1}(1) > \cdots < g^{m_N}(N) > ,$$

where the m_r, $1 \leq r \leq N$, are arbitrary non-negative integers subject to the condition $m_1 + m_2 + \cdots + m_N = 2n$. Since all the moments of $\hat{\beta}(r)$, $g(r)$, $1 \leq r \leq N$, are non-negative, the desired inequality (31) follows at once from a term-by-term comparison of (32) and (33), together with the obvious inequalities

$$< \hat{\beta}^{2n} > = A^{2n} \leq 1 \cdot 3 \cdots (2n-1)A^{2n} = < g^{2n} >,$$

$$< \hat{\beta}^{2n+1} > = 0 \leq 0 = < g^{2n+1} > \qquad , \quad 0 \leq n < \infty ,$$

where $\hat{\beta}$ denotes any of the $\hat{\beta}(r)$ and g denotes any of the $g(r)$.

Applying the DeMoivre-Laplace limit theorem [3] (the special case of the central limit theorem appropriate to a sum of independent, identically

distributed Bernoulli random variables), we would expect to find that $\hat{\beta}_N$ is approximately Gaussian, with mean zero and variance A^2N, to an approximation which improves as N increases. The difference between the distribution of $\hat{\beta}_N$ and that of g_N is greatest in the tails of the distributions; in particular, the values of $\hat{\beta}_N$ are confined to the interval (-AN, +AN), whereas g_N takes on values outside any finite interval. Thus, for fixed N, we can expect $< g_N^{2n} >$ to become a poorer approximation to $< \hat{\beta}_N^{2n} >$ as n increases. In fact, for large n we can use Stirling's formula

$$n! \sim \sqrt{2\pi}\ n^{n+\frac{1}{2}}\ e^{-n}$$

to derive the approximation

$$< g_N^{2n} > \sim \sqrt{2}\ \left(\frac{2nA^2N}{e}\right)^n . \tag{34}$$

It follows that ultimately $< g_N^{2n} >$ goes to infinity as n increases, whereas, as already noted, $< \hat{\beta}_N^{2n} >$ must go to zero as n increases. (We assume, of course, that the condition AN < 1 is satisfied.) Using (34), we easily see that the minimum value of $< g_N^{2n} >$ is obtained for

$$2n_1 \sim \frac{1}{A^2N} , \tag{35}$$

and is approximately $\sqrt{2}\ \exp(-n_1)$.

These properties of the Gaussian estimate of $< \hat{\beta}_N^{2n} >$ are illustrated in Figure 2, where we have plotted $< \hat{\beta}_N^{2n} >$ and $< g_N^{2n} >$ for the case N = 20, AN = 0.99. It will be noted that the Gaussian estimate of $< \hat{\beta}_N^{2n} >$ is good for $n \leq 10$, and then deteriorates rapidly. The quantity $< g_N^{2n} >$ is a minimum for $2n_1 = 20 \sim (A^2N)^{-1}$, in agreement with (35); it then increases rapidly and ultimately diverges.

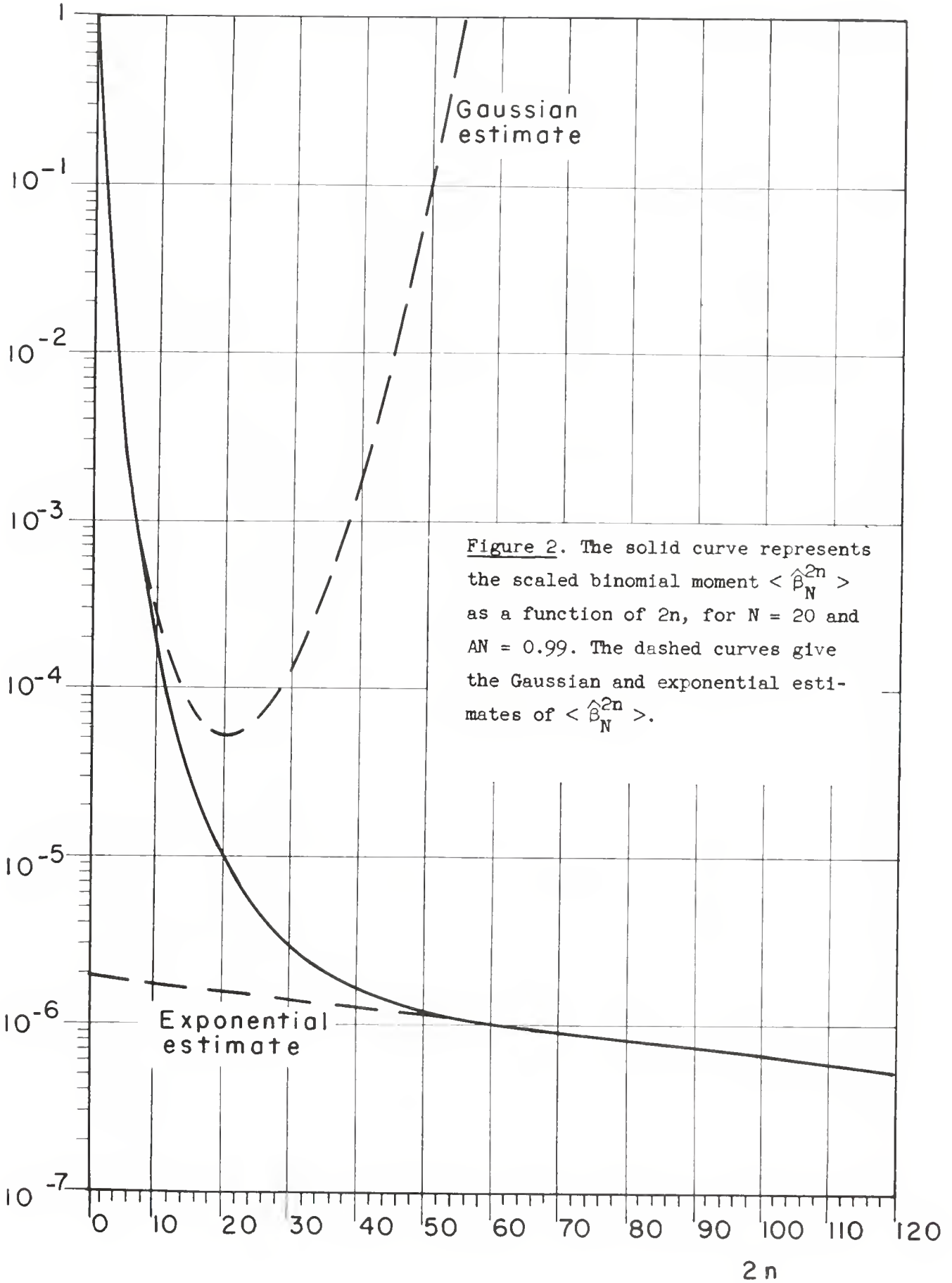

Figure 2. The solid curve represents the scaled binomial moment $< \hat{\beta}_N^{2n} >$ as a function of 2n, for N = 20 and AN = 0.99. The dashed curves give the Gaussian and exponential estimates of $< \hat{\beta}_N^{2n} >$.

b. The exponential estimate of $< \hat{\beta}_N^{2n} >$

When $2n \gg N$, we have the approximation

$$< \hat{\beta}_N^{2n} > \sim 2^{1-N}(AN)^{2n},$$

which improves as n increases. To see this, we examine the terms of the sum (29) in the limit as $n \to \infty$. Thus, for example, the ratio of the tail term corresponding to $r = 0$ to the term corresponding to $r = 1$ is

$$\frac{1}{N}\left(\frac{N}{N-2}\right)^{2n},$$

which goes to infinity as $n \to \infty$. Applying this argument to the other terms in the sum (29), it is clear that as n increases, most of the contribution to $< \hat{\beta}_N^{2n} >$ comes from the two tail terms corresponding to $r = 0$ and $r = N$, which in the limit of infinite n completely dominate the other terms of the sum (29). It follows that for large n

$$< \hat{\beta}_N^{2n} > \sim 2^{-N}\binom{N}{0}(AN)^{2n} + 2^{-N}\binom{N}{N}(-AN)^{2n} = 2^{1-N}(AN)^{2n},$$

as asserted. In particular, the condition $AN < 1$ is still needed to ensure the convergence of the stochastic majorants.

The exponential estimate $2^{1-N}(AN)^{2n}$ is drawn in Figure 2 for the case $N = 20$, $AN = 0.99$. We see that in this case $< \hat{\beta}_N^{2n} >$ becomes asymptotic to its exponential estimate for $2n \gtrsim 70$. The portion of the $< \hat{\beta}_N^{2n} >$ vs. 2n curve for which neither the Gaussian estimate nor the exponential estimate is good must be filled in by using (29).

Tables 1 and 2 give values of $< \hat{\beta}_N^{2n} >$, in the ranges where they are appreciable, for $N = 20, 35, 50$ and $AN = 0.50, 0.99$. The use of such tables in estimating the strength of multiple scattering is discussed in the next section.

2n	N = 20	N = 35	N = 50	2n	N = 20
2	.049005000 (.049005000)	.028002857 (.028002857)	.019602000 (.019602000)	38	.000001830
4	.006964321 (.007204470)	.002307671 (.002352480)	.001137346 (.001152715)	40	.000001686
6	.001593455 (.001765275)	.000310846 (.000329381)	.000108507 (.000112978)	42	.000001571
8	.000492723 (.000605551)	.000057478 (.000064565)	.000014296 (.000015502)	44	.000001476
10	.000188968 (.000267075)	.000013395 (.000016272)	.000002389 (.000002735)	46	.000001398
12	.000085391 (.000143968)	.000003740 (.000005012)	.000000481 (.000000590)	48	.000001332
14	.000043934 (.000091717)	.000001209 (.000001825)	.000000113 (.000000150)	50	.000001276
16	.000025112 (.000067419)	.000000442 (.000000766)	.000000030 (.000000044)	52	.000001227
18	.000015654 (.000056166)	.000000179 (.000000365)	.000000009 (.000000015)	54	.000001185
20	.000010490 (.000052296)	.000000080 (.000000019)	.000000003 (.000000005)	56	.000001147
22	.000007469	.000000038	.000000001	58	.000001113
24	.000005597	.000000020		60	.000001081
26	.000004380	.000000011	less	62	.000001053
28	.000003555	.000000006	than	64	.000001026
30	.000002977	.000000004	10^{-9}	66	.000001001
32	.000002559	.000000002		68	.000000978
34	.000002248	.000000002		70	.000000956
36	.000002013	.000000001		72	.000000934

Table 1. The scaled binomial moments $< \hat{\beta}_N^{2n} >$ for the case AN = 0.99 and various values of N. The numbers in parentheses are the corresponding Gaussian estimates of $< \hat{\beta}_N^{2n} >$.

2n	N = 20	N = 35	N = 50
2	.012500000 (.012500000)	.007142857 (.007142857)	.005000000 (.005000000)
4	.000453125 (.000468750)	.000150146 (.000153061)	.000074000 (.000075000)
6	.000026445 (.000029297)	.000005159 (.000005466)	.000001801 (.000001875)
8	.000002086 (.000002563)	.000000243 (.000000273)	.000000061 (.000000066)
10	.000000204 (.000000288)	.000000014 (.000000018)	.000000003 (.000000003)
12	.000000024 (.000000040)	.000000001 (.000000001)	less than 10^{-9}
14	.000000003 (.000000006)	less than 10^{-9}	

Table 2. The scaled binomial moments $< \hat{\beta}_N^{2n} >$ for the case AN = 0.50 and various values of N. The numbers in parentheses are the corresponding Gaussian estimates of $< \hat{\beta}_N^{2n} >$.

5. Analysis of multiple scattering

The application of the results obtained in the preceding section to the problem of multiple scattering in a binomially distributed, random stack of dielectric slabs is straightforward since the terms of the stochastic majorant series (24) and (28) are just the binomial moments $< \hat{\beta}_N^{2n} >$ multiplied by the growth factors 2n+1 and 2n-1.* To illustrate the situation, we plot in Figure 3 the general term $(2n-1) < \hat{\beta}_N^{2n} >$ of the stochastic majorant series (28) for $|E_R(2n_o)|$, for the case $AN = 0.99$, $N = 20$. Comparing Figures 2 and 3, we see that in both cases the terms at first fall off rapidly with increasing n, then fall off less rapidly as 2n approaches N, and finally approach an asymptotic curve corresponding to the exponential estimate of $< \hat{\beta}_N^{2n} >$. The exponential estimate of the general term of the stochastic majorant series (24) is given by

$$(36) \qquad (2n-1) < \hat{\beta}_N^{2n} > \sim (2n-1)2^{1-N} (AN)^{2n}.$$

The quantity (36) is a maximum for

$$2n-1 \sim - \frac{1}{\log_e AN} .$$

For the case $AN = 0.99$, this maximum occurs at $2n-1 \sim 100$ (see Figure 3).

For comparison, we also plot in Figure 3 the terms of the sure majorant series (18), which, it will be recalled, differ from the corresponding terms of the stochastic majorant series (28) by having $(AN)^{2n}$ in place of $< \hat{\beta}_N^{2n} >$. Thus, except for the factor 2^{1-N}, the exponential estimate of the terms of the stochastic majorant series is the same as for the sure majorant series. This is apparent from Figure 3, where the curves representing the terms of the sure

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* These factors represent the number of terms in the Neumann series for $T(\alpha)$ and $R(\alpha)$ which combine to give the term of order 2n in the Neumann series for $<|R|^2>$ and $<|T|^2>$.

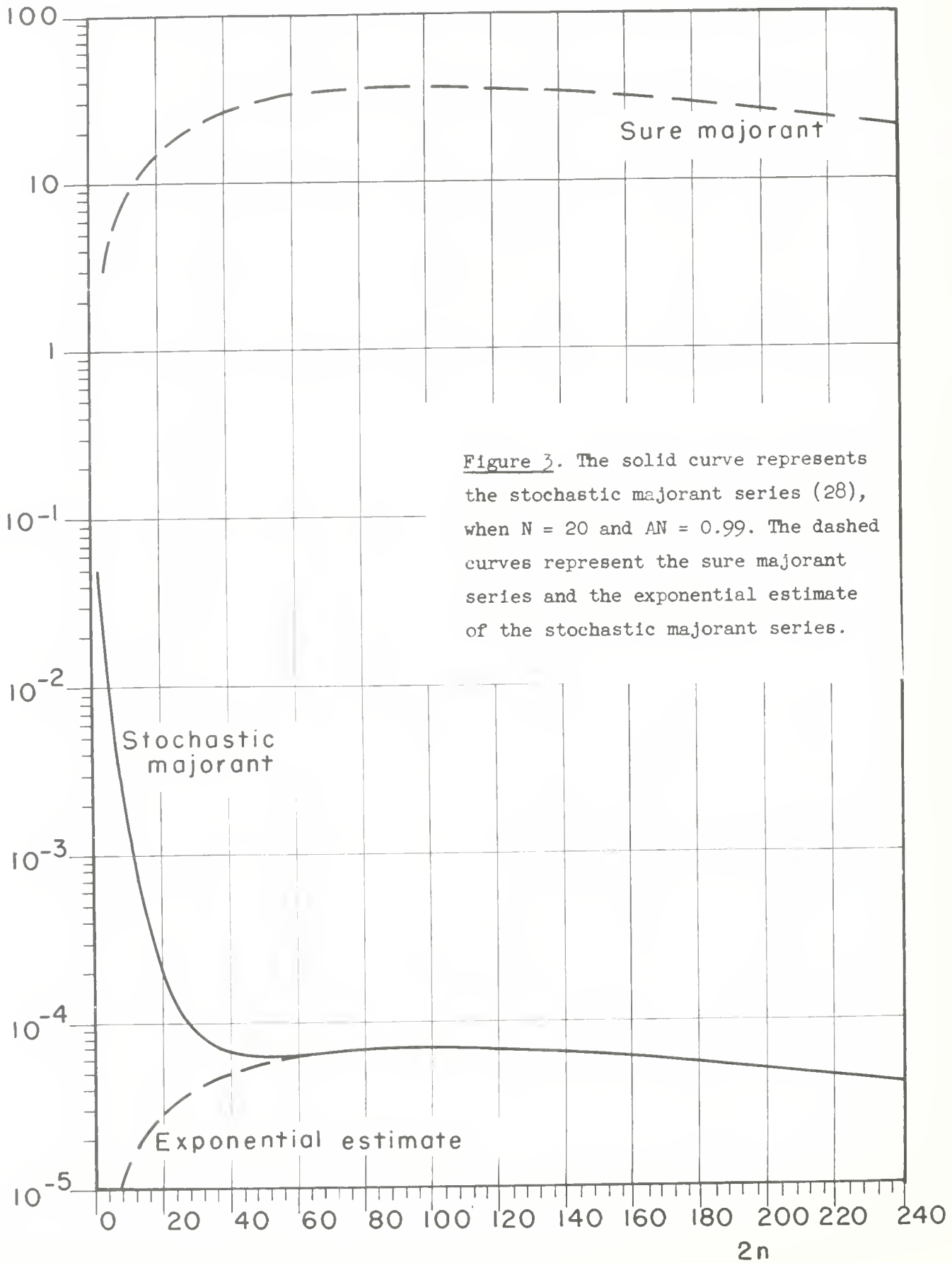

Figure 3. The solid curve represents the stochastic majorant series (28), when N = 20 and AN = 0.99. The dashed curves represent the sure majorant series and the exponential estimate of the stochastic majorant series.

majorant series and the exponential estimate of the terms of the stochastic majorant series differ only by a vertical translation. The appearance of the terms of the stochastic majorant series (24) for $|E_T(2n_o)|$ differs only slightly from that of the stochastic majorant series (28) for $|E_R(2n_o)|$, due to the closeness of the growth factors 2n-1 and 2n+1 for large n.

It is now a simple matter to estimate the error committed in neglecting multiple scattering greater than a given order. We illustrate the general procedure by two examples.

a. Mean square reflection coefficient; AN = 0.99, N = 20

As we have seen in Section 4, the absolute value of the error $E_R(2n_o)$ committed by neglecting the terms of order $2n_o$ and greater in the Neumann series (15) for $<|R|^2>$ is bounded above by the sum

$$(37) \qquad \sum_{n=n_o}^{\infty} (2n-1) < \hat{\beta}_N^{2n} >.$$

To evaluate (37) for various values of n_o, we use the data of Table 2, together with the estimate

$$(38) \qquad \sum_{n=35}^{\infty} (2n-1) < \hat{\beta}_N^{2n} > \sim 2^{1-N} \sum_{n=35}^{\infty} (2n-1)(AN)^{2n}$$

The use of (38) is justified by the accuracy of the exponential estimate of $< \hat{\beta}_N^{2n} >$ for $2n \gtrsim 70$. It is not hard to see that the right side of (38) equals

$$2^{1-N}(AN)^{70} \quad \frac{69-67(AN)^2}{(1-(AN)^2)^2} = .000794$$

whence $|E_R(2n_o)|$ is bounded above by*

$$(39) \qquad B(2n_o) \sim \sum_{n=n_o}^{34} (2n-1) < \hat{\beta}_N^{2n} > + .000794.$$

---

* To ensure that (39) is an upper bound, one could multiply the second term in the right side of (39) by 1.1, say. This would not affect the data of Table 3 to the accuracy given.

The data presented in Table 3 was obtained by using (39).

It is easy to calculate explicitly the first term R_2 of the Neumann series (15) for $\langle|R|^2\rangle$, since it is just the mean square absolute value of the random variable

$$\frac{ik}{2}\int_0^{Nd} e^{2ikz}\,\Delta\epsilon(z)\,dz. \tag{40}$$

The integral in (40) reduces to

$$\sum_{r=1}^{N}\Delta\epsilon(r)\int_{(r-1)d}^{rd} e^{2ikz}dz = \frac{\sin kd}{k}\sum_{r=1}^{N}\Delta\epsilon(r)e^{ik(r-1)d},$$

whence

$$R_2 = A^2N\left(\frac{\sin kd}{kd}\right)^2. \tag{41}$$

On the other hand, the stochastic majorant for R_2 is, of course, the quantity A^2N. It follows that R_2 is close to its stochastic majorant if $kd \ll 1$, i.e., if the slab width d is small compared to the wavelength of the incident radiation. Since in the present case $A^2N = 0.049005$, Table 3 shows that the first term in (15) and the absolute value of the sum of the next few terms may well be of comparable size. Indeed, if kd is large, we cannot even guarantee that single scattering predominates.

b. Mean square transmission coefficient; $AN = 0.50$, $N = 250$, $A^2N = 10^{-3}$

We begin by showing that the second term T_2 of the Neumann series (14) for $\langle|T|^2\rangle$ is non-positive. It follows from (6) that T_2 is the sum of the two terms

$$\frac{k^2}{4}\left\langle\left|\int_0^{Nd}\Delta\epsilon(z)\,dz\right|^2\right\rangle \tag{42}$$

and

$2n_o$	$B(2n_o)$
4	.039
6	.018
8	.010
10	.007
20	.003
30	.002
40	.002
50	.001

Table 3. $B(2n_o)$ is an upper bound for the absolute value of the error committed by neglecting terms of order $\geq 2n_o$ in the Neumann series (15) for $\langle |R|^2 \rangle$, for the case AN = 0.99, N = 20.

$$(43) \qquad -\frac{k^2}{4}\left\langle \int_0^{Nd}\int_0^{Nd} e^{-ikz_1}\, e^{ik|z_1-z_2|}\, e^{ikz_2}\, \Delta\epsilon(z_1)\Delta\epsilon(z_2)dz_1dz_2 \right\rangle$$

$$+ \text{ complex conjugate.}$$

Since

$$\int_0^{Nd} \Delta\epsilon(z)dz = d\sum_{r=1}^{N} \Delta\epsilon(r),$$

(43) reduces to

$$(44) \qquad \frac{k^2d^2}{4}\left\langle \sum_{r=1}^{N} \Delta\epsilon(r) \sum_{s=1}^{N} \Delta\epsilon(s) \right\rangle = A^2N.$$

The integral in (43) need only be evaluated for the case where z_1 and z_2 both lie in the same interval from (r-1)d to rd, since otherwise the integral vanishes. Doing the integration, we find that (43) reduces to

$$(45) \qquad -\frac{k^2}{4}\left[\frac{d^2}{2} - \frac{d}{2ik} - \frac{1}{4k^2}\left(e^{2ikd}-1\right)\right]\left\langle \sum_{r=1}^{N} \left(\Delta\epsilon(r)\right)^2 \right\rangle + \text{ complex conjugate}$$

$$= -A^2N - A^2N\left(\frac{\sin kd}{kd}\right)^2.$$

Combining (44) and (45), we obtain

$$T_2 = -A^2N\left(\frac{\sin kd}{kd}\right)^2$$

which is non-positive. The quantity T_2 is bounded in absolute value by its stochastic majorant A^2N, and is approximately $-A^2N$ for small values of kd. Furthermore, we note that the sum of T_2 and the quantity R_2 given by (41) vanishes, a result which might be expected, since the sum of $\langle|T|^2\rangle$ and $\langle|R|^2\rangle$ must be unity by energy conservation.

We now estimate the error committed when we approximate the mean square transmission coefficient by

$$(46) \qquad \langle |T|^2 \rangle \sim 1 + T_2 = 1 - A^2N \left(\frac{\sin kd}{kd}\right)^2 = 1 - 10^{-3} \left(\frac{\sin kd}{kd}\right)^2,$$

thereby neglecting all terms of order ≥ 4 in the Neumann series (14). To do this, we first sum the terms of order $\geq 2\nu$ in the sure majorant series (17) for $\langle |T|^2 \rangle$, obtaining

$$(47) \qquad \sum_{n=\nu}^{\infty} (2n+1)(AN)^{2n} = \frac{(AN)^{2\nu}}{\left(1-(AN)^2\right)^2} \left[(2\nu+1)+(1-2\nu)(AN)^2\right].$$

For the value AN = 0.5 under consideration, and for $\nu = 15$, this sum is approximately 10^{-8}, which is an upper bound, albeit a very crude one, for the terms of order ≥ 30, which were omitted in writing (46). The rest of the terms omitted in writing (46), with orders from 4 to 28, are bounded above by the series

$$(48) \qquad \sum_{n=2}^{14} (2n+1) \langle g_N^{2n} \rangle,$$

i.e., by the Gaussian estimate of the corresponding terms of the stochastic majorant series (24), which, since N = 250, should be quite a good estimate. Using (30), and then adding (47) and (48), we easily find that for the case $A^2N = 10^{-3}$ the absolute value of the error committed in using the approximation (46) is less than 1.6×10^{-5}, say, which is only a small percentage of $|T_2|$ if $kd \ll 1$.

6. Conclusion

We have seen that the absolute values of successive terms in the Neumann series (14) and (15) for the mean square transmission coefficient and the mean square reflection coefficient may fall off rapidly with increasing n, even if the convergence parameter AN is close to unity. Indeed, an examination of the formula (30) for the Guassian estimates of the binomial moments shows that the smallness of $A\sqrt{N}$ (rather than of AN) is enough to ensure that the terms of the series (14) and (15) fall off rapidly in absolute value, at least until the order of the terms is comparable with the number of slabs N. In particular,

if $kd < \pi$, then for sufficiently small $A\sqrt{N}$, the formulas

$$< |R|^2 > \sim R_2, \quad < |T|^2 > \sim 1 + T_2,$$

corresponding to the neglect of terms of order higher than two in the Neumann series (14) and (15), will be good approximations, even if AN is close to unity. (Note that $AN \sim 1$ and $A\sqrt{N} \ll 1$ imply that $N \gg 1$.) One might adduce this example in support of the general belief that in the problem of radio scattering beyond the horizon, the large number of "scattering blobs" within the common volume of the antenna beams helps to justify the conventional use of the single scattering approximation for the scattered field.

When the order of the terms is greater than N, we find that we must again use the size of AN, rather than the size of $A\sqrt{N}$, to estimate the rate of convergence of the series for $< |T|^2 >$ and $< |R|^2 >$. Qualitatively speaking, as n increases, the random nature of the scattering medium becomes progressively less effective in attenuating scattering of order n; finally, when n becomes large compared to the number of slabs, the randomness of the medium plays no further role in attenuating multiple scattering.

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